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RESOLUTION METHODS AND APPLIED PROBLEMS OF GAME THEORY

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1. Methods for Solving Matrix Games

Let the game involve two parties *A* and *B*. The playing field is given by the payoff matrix (payment matrix – table 1):

Table 1

A		B				
		B_1	...	B_j	...	B_n
		x_1	...	x_j	...	x_n
A_1	y_1	a_{11}	...	a_{ij}	...	a_{1n}
...	...	...	...	...	...	...
A_i	y_i	a_{i1}	...	a_{ij}	...	a_{in}
...	...	...	...	...	...	...
A_m	y_m	a_{m1}	...	a_{mj}	...	a_{mn}

The strategy chosen by the party *A*, will be denoted as $A_1, A_2, \ldots, A_t$; and side *B* strategy will be given as $B_1, B_2, \ldots$, $\boldsymbol{B_n}$;$\boldsymbol{y_i}$ – probability of strategy use by the first party; x_j – the probability of using the *j* trategy by the second party *B*. *A vector is the first (second) player's mixed strategy*

$$\bar{\boldsymbol{y}} = (\boldsymbol{y_1}, \ldots, \boldsymbol{y_m}), \bar{\boldsymbol{x}} = (\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}),$$

for which

$$\sum_{i=1}^{m} y_1 = 1;\ \sum_{j=1}^{n} x_i = 1;\ y_i \geq 0(i = \overline{1,m});\ x_j \geq 0(j = \overline{1,n}).$$

Elements of the payoff matrix can be positive, negative, or equal to zero. If the element of the matrix is positive, then party *B* in a certain situation pays the party *A* a sum of money equal to the element of the matrix.

If the element of the matrix is negative, then party *A* pays party *B* a um of money equal to the absolute value of the element. If the element is zero, no payment is made.

We will consider *zero-sum paired games*[1].

These are games whose payment amount is zero, that is, the loss of one player is equal to the win of another. In this case, the average gain (loss) – a mathematical expectation is a function of mixed strategies $\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}$:

Function *S (x,* y) is called *a payment function* of the game with matrix $[\boldsymbol{a_{ij}}]_{\boldsymbol{mxn}}$.

Strategies $\overline{\boldsymbol{y}^*} = (\boldsymbol{y}_1^*, \dots, \boldsymbol{y}_m^*), \overline{\boldsymbol{x}^*} = (\boldsymbol{x}_1^*, \dots, \boldsymbol{x}_n^*)$ are called *optimal,* if for the random strategies $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_m), \overline{\boldsymbol{x}} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ these requirements are satisfied

$$S(\bar{y}, \overline{x^*}) \le S(\overline{y^*}, \overline{x^*}) \le S(\overline{y^*}, \bar{x}). \quad (1)$$

Using the optimal mixed strategies $\overline{\boldsymbol{y}^*}, \overline{\boldsymbol{x}^*}$in game gives the first player a win no less than while using any other strategy $\overline{\boldsymbol{y}}$;and gives the second player a loss no bigger than while using any other strategy $\overline{\boldsymbol{x}}$.

The value of the payment function with optimal strategies determines *the price of the game C,* i.e $\boldsymbol{C} = \boldsymbol{S}(\overline{\boldsymbol{y}}^*, \overline{\boldsymbol{x}}^*)$

The combination of optimal strategies and the price of the game is the solution of the game.

It is proved that in order for the number C to be the price of the game, and $\overline{\boldsymbol{y}}^*$ and $\overline{\boldsymbol{x}}^*$ to be optimal strategies, it is necessary and sufficient the inequalities to work

$$\textstyle\sum_{i=1}^{m} a_{ij}\, y_i^* \ge C(j = \overline{1, n});\ \sum_{j=1}^{n} a_{ij} x_j^* \le C(i = \overline{1, m}). \quad (2)$$

In the future, for certainty, assume that $\mathrm{C} > \boldsymbol{0}$. This can always be achieved by that the adding to all elements of the payoff matrix the same constant number *d* does not change the optimal strategies, but only increases the price of the game for *d*.

[1] Neumann D., Morgenstern O. Theory of Games and Economic behavior. Moskow: Science, 1970, 708 p.

1.1 Reduction of problems of theory of games to problems of linear programming By dividing both parts of the first of inequalities (15) by C, we get the system in the expanded form[2]:

$$\begin{cases} \boldsymbol{a_{11}y'_1 + a_{21}y'_2 + \cdots + a_{m1}y'_m \geq 1,} \\ \boldsymbol{a_{12}y'_1 + a_{22}y'_2 + \cdots + a_{m2}y'_m \geq 1,} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \boldsymbol{a_{1n}y'_1 + a_{2n}y'_2 + \cdots + a_{m2}y'_m \geq 1,} \end{cases} \quad (3)$$

where $\boldsymbol{y'_1 = \frac{y_i^*}{C} (i = \overline{1,m}); \ y'_1 \geq 0}$.

Using the last notation, condition $\boldsymbol{\sum_{i=1}^{m} y_i^* = 1}$ can be written as $\boldsymbol{\sum_{i=1}^{m} y_i'^* = \frac{1}{C}}$

As the first player tries to get the maximum win, he must provide a minimum value of 1/ *C*. With this in mind, determining the optimal strategy for the first player comes down to finding the minimum value of the function

$$\mathbf{f = \sum_{i=1}^{m} y'_i = y'_1 + y'_2 + \cdots + y'_m} \quad (4)$$

under conditions (16).

Similar considerations show that determining the optimal second player's strategy comes down to finding the maximum value of the function

$$\boldsymbol{F = \sum_{j=1}^{n} x'_i = x'_1 + x'_2 + \cdots + x'_n} \quad (5)$$

under conditions

$$\begin{cases} \boldsymbol{a_{11}x'_1 + a_{12}x'_2 + \cdots + a_{1n}x'_n \leq 1,} \\ \boldsymbol{a_{21}x'_1 + a_{22}x'_2 + \cdots + a_{2n}x'_n \leq 1,} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \boldsymbol{a_{m1}x'_1 + a_{m2}x'_2 + \cdots + a_{2n}x'_n \leq 1,} \\ \boldsymbol{x'_j \geq 0 \ (j = \overline{1,n})} \end{cases}$$

where $\boldsymbol{x'_j = x_j^*/C}$.

Thus, in order to find the solution of the game given by this payment matrix (see table. 1), it is necessary to make dual (conjugated) linear programming problems and solve them.

[2] Akulich I.L. Mathematical programming in problem examples. Moskow: Higher school, 1986, 318 p.

The straightforward problem is to find the maximum value of the function *F*, given by expression (5) under conditions (6).

Dual (conjugate) problem is find the minimum value of function$\boldsymbol{f}$given by expression (4) under condition (3).

Using a solution of a pair of dual problems

$$\overline{\boldsymbol{y}'^{*}} = (\boldsymbol{y}_{\boldsymbol{1}}'^{*}, \dots, \boldsymbol{y}_{\boldsymbol{m}}'^{*}), \overline{\boldsymbol{x}_{\boldsymbol{1}}'^{*}} = (\boldsymbol{x}_{\boldsymbol{1}}', \dots, \boldsymbol{x}_{\boldsymbol{n}}'^{*}), \quad (6)$$

we get formulas for determining strategies and the price of the game:

$$\boldsymbol{y}_{\boldsymbol{i}}^{*} = \frac{\boldsymbol{y}_{\boldsymbol{i}}'^{*}}{\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{m}} \boldsymbol{y}_{\boldsymbol{i}}'^{*}} = \boldsymbol{C}\boldsymbol{y}_{\boldsymbol{i}}'^{*}; \boldsymbol{x}_{\boldsymbol{j}}'^{*} = \frac{\boldsymbol{x}_{\boldsymbol{j}}^{*}}{\sum_{\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}} \boldsymbol{x}_{\boldsymbol{j}}'^{*}} = \boldsymbol{C}\boldsymbol{x}_{\boldsymbol{j}}'^{*}, \quad (7)$$

$$\boldsymbol{C} = \frac{\boldsymbol{1}}{\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{m}} \boldsymbol{y}_{\boldsymbol{i}}'^{*}} = \frac{\boldsymbol{1}}{\sum_{\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}} \boldsymbol{x}_{\boldsymbol{j}}'^{*}}. \quad (8)$$

So, the process of finding a solution to the game using linear programming methods involves the following steps:

1. Assembling of a pair of dual (conjugate) linear programming problems that are equivalent to such a matrix game.
2. Determining optimal plans for dual problems.
3. Finding a solution to the game, using the relationship between dual problems' plans, optimal strategies and the price of the game.

According to these steps, we will solve the above-mentioned problem of supply of raw materials by linear programming methods. In this problem (game) the payment matrix is given in Table 2. In order for the price of game C to be greater than zero, we add the number d = 400 to all elements of this matrix. This, as mentioned above, will not change the optimal strategies, but will only increase the price of the game by d = 400. After that adding a payment matrix will look like

$$A = \begin{pmatrix} 300 & 0 \\ 250 & 100 \\ 210 & 150 \\ 70 & 200 \end{pmatrix}.$$

According to the first stage, we make a pair of dual (conjugate) linear programming problems that are equivalent to a given matrix game.

Direct problem (relations (5), (6)) is to find the maximum value of the function

$$F = \sum_{j=1}^{2} x_i' = x_1' + x_2' (n = 2; m = 4) \quad (9)$$

with restrictions

$$\begin{cases} \mathbf{300}\boldsymbol{x}'_\mathbf{1} \le \mathbf{1}, \\ \mathbf{250}\boldsymbol{x}'_\mathbf{1} + \mathbf{100}\boldsymbol{x}'_\mathbf{2} \le \mathbf{1}, \\ \mathbf{210}\boldsymbol{x}'_\mathbf{1} + \mathbf{150}\boldsymbol{x}'_\mathbf{2} \le \mathbf{1}, \\ \mathbf{70}\boldsymbol{x}'_\mathbf{1} + \mathbf{200}\boldsymbol{x}'_\mathbf{2} \le \mathbf{1}, \\ \boldsymbol{x}'_\mathbf{1}, \boldsymbol{x}'_\mathbf{2} \ge \mathbf{0}. \end{cases} \tag{10}$$

Dual (conjugate) problem (relations (16) and (17)) is to find the minimum value of the function

$$f = \sum_{i=1}^{4} y'_i = y'_1 + y'_2 + y'_3 + y'_4 \tag{11}$$

with restrictions

Having solved the problems of linear programming (9) – (12) by the simplex method, we obtain

$$\boldsymbol{x}'^*_\mathbf{1} = \mathbf{5/3150}\,;\ \boldsymbol{x}'^*_\mathbf{2} = \mathbf{14/3150}\,; \boldsymbol{F}_{\boldsymbol{max}} = \mathbf{19/3150};$$

$$\boldsymbol{y}'^*_\mathbf{1} = \mathbf{0};\ \boldsymbol{y}'^*_\mathbf{2} = \mathbf{0}; \boldsymbol{y}'^*_\mathbf{3} = \mathbf{13/3150}\,; \boldsymbol{y}'^*_\mathbf{4} = \mathbf{6/3150}; \boldsymbol{f}_{\boldsymbol{min}} = \mathbf{19/3150}.$$

Substituting these solutions into relations (20) and (21), we obtain the optimal strategies of the firm *A:*

$$\boldsymbol{y}^*_\mathbf{1} = \frac{\boldsymbol{y}'^*_\mathbf{1}}{\sum_{\boldsymbol{i}=\mathbf{1}}^{\mathbf{4}} \boldsymbol{y}'^*_{\boldsymbol{i}}} = \frac{\mathbf{0}}{\frac{\mathbf{19}}{\mathbf{3150}}} = \mathbf{0};\ \boldsymbol{y}^*_\mathbf{2} = \frac{\boldsymbol{y}'^*_\mathbf{2}}{\sum_{\boldsymbol{i}=\mathbf{1}}^{\mathbf{4}} \boldsymbol{y}'^*_{\boldsymbol{i}}} = \frac{\mathbf{0}}{\frac{\mathbf{19}}{\mathbf{3150}}} = \mathbf{0};$$

$$\boldsymbol{y}^*_\mathbf{3} = \frac{\boldsymbol{y}'^*_\mathbf{3}}{\sum_{\boldsymbol{i}=\mathbf{1}}^{\mathbf{4}} \boldsymbol{y}'^*_{\boldsymbol{i}}} = \frac{\frac{\mathbf{13}}{\mathbf{3150}}}{\frac{\mathbf{19}}{\mathbf{3150}}} = \mathbf{0{,}685};\ \boldsymbol{y}^*_\mathbf{4} = \frac{\boldsymbol{y}'^*_\mathbf{4}}{\sum_{\boldsymbol{i}=\mathbf{1}}^{\mathbf{4}} \boldsymbol{y}'^*_{\boldsymbol{i}}} = \frac{\frac{\mathbf{6}}{\mathbf{3150}}}{\frac{\mathbf{19}}{\mathbf{3150}}} = \mathbf{0{,}315},$$

optimal strategies of the supplier company ***B:***

$$\boldsymbol{x}^*_\mathbf{1} = \frac{\boldsymbol{x}'^*_\mathbf{1}}{\sum_{\boldsymbol{j}=\mathbf{1}}^{\mathbf{2}} \boldsymbol{x}'^*_{\boldsymbol{j}}} = \frac{\frac{\mathbf{5}}{\mathbf{3150}}}{\frac{\mathbf{19}}{\mathbf{3150}}} = \mathbf{0{,}263};\ \boldsymbol{x}^*_\mathbf{2} = \frac{\boldsymbol{x}'_\mathbf{2}}{\sum_{\boldsymbol{j}=\mathbf{1}}^{\mathbf{2}} \boldsymbol{x}'^*_{\boldsymbol{j}}} = \frac{\frac{\mathbf{14}}{\mathbf{3150}}}{\frac{\mathbf{19}}{\mathbf{3150}}} = \mathbf{0{,}737}$$

and the price of the game

$$\boldsymbol{C} = \frac{\mathbf{1}}{\sum_{\boldsymbol{i}=\mathbf{1}}^{\mathbf{4}} \boldsymbol{y}'^*_{\boldsymbol{i}}} = \frac{\mathbf{1}}{\sum_{\boldsymbol{j}=\mathbf{1}}^{\mathbf{2}} \boldsymbol{x}'^{\boldsymbol{x}}_{\boldsymbol{j}}} = \frac{\mathbf{1}}{\frac{\mathbf{19}}{\mathbf{3150}}} \approx \mathbf{165{,}8}.$$

Since adding to all elements of the payment matrix the number d = 400 has increased the price of the game by 400, the true price of the game of the initial problem (expected losses of the firm *A)* will be

165.8 – 400 = -234.2 $

As it is easy to check, the optimal strategies and the price of the game found by linear programming methods are exactly the same as those found above using the graphical method.

Unlike the graphical method that can be applied when either $\boldsymbol{n} \leq \mathbf{2}$ or $\boldsymbol{m} \leq \mathbf{2}$, the linear programming method can be applied to arbitrary finite values *mi n*

1.2 An iterative (approximate) method for solving the problems of game theory Two approaches to solving the problems of game theory have been considered above: graphic and reduction to linear programming problems. In both cases there is an exact solution to the problems of game theory – the price and optimal mixed strategies of players A and B.

Let us now consider an approximate method for solving the problems of game theory, which reflects to some extent the real situation of the players' gradual accumulation of experience in adopting rational strategies as a result of many repetitions of conflict situations (games)[3].

This method allows you to simulate the process of training (behavior) of players during the repetition of the game, when each of them evaluates the behavior of the opponent and responds to it in the best way for themselves. Each time at the beginning of the game, they choose the most advantageous strategies for themselves, basing on the previous choices of the opponent.

Let us solve, using this method, the previous problem with firms *A* and *B,* for which the payment matrix is given in Table 2 in the case when the game is antagonistic.

On the first day after the conclusion of the contract, firms A and B accept random strategies, for example: firm A uses strategy A_3 (–190, –250), firm *B* uses strategy B_2 (–400, –300, –250, –200).

Let us build a model that describes the rules for choosing the next «moves» by firms *A* and *B.*

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[3] Kudryavtsev E.M. Research of operations in problems, algorithms and programs. Moskow: Radio Communication, 1984, 184 p.

On the second day, the firm *A* chooses its strategy so that its win with the strategy B_2 of the company *B* was the maximum, i.e the losses, taking into account the signs of payment, were minimal (–200). Obviously, this will be the strategy A_4 (–330, –200).

Firm *B,* taking into account the previous day, chooses the strategy B_2 again to inflict the firm *A* with the greatest losses (-250) when its strategy is A_3.

On the third day, the firm *A* chooses its strategy so that its accumulated (total) losses for the previous two days with the strategies B_2 of the firm *B*

$$(\boldsymbol{S_{A_1}}, \boldsymbol{S_{A_2}}, \boldsymbol{S_{A_3}}, \boldsymbol{S_{A_4}}) = (\mathbf{-400}, \mathbf{-300}, \mathbf{-250}, \mathbf{-200}) +$$
$$+(\mathbf{-400}, \mathbf{-300}, \mathbf{-250}, \mathbf{-200}) = (\mathbf{-800}, \mathbf{-600}, \mathbf{-500}, \mathbf{-400})$$

were minimal (they are highlighted). Obviously, this will be the strategy A_4. Firm*B* selects its strategy on the same day, based on information on the strategies of the firm ***A*** for the previous two days, so that the total losses of the firm ***A*** with its strategies $\boldsymbol{A_3}$ $\boldsymbol{iA_4}$,

$$(\boldsymbol{S_{B_1}}, \boldsymbol{S_{B_2}}) = (\mathbf{-190}, \mathbf{-250}) + (\mathbf{-330}, \mathbf{-200}) = (\mathbf{-520}, \mathbf{-450}),$$

were maximal (they are highlighted). This is $\boldsymbol{B_1}$ strategy

On the fourth day, the situation is repeated. Firm *A,* Basing on the previous actions of the firm *B,* in three days chooses its strategy so that its total losses for these days with the strategies B_2, B_2, B_1 of the firm *B,*

$$(S_{A_1}, S_{A_2}, S_{A_3}, S_{A_4}) = (-800, -600, -500, -400) +$$
$$+(-100, -150, -192, -330) = (-900, -750, -690, -730),$$

were minimal. This is strategy A_3.

Firm *B,* whose purpose is to maximize the losses of the firm *A* with its strategies A_3, A_a,A_4,

$$(\boldsymbol{S_{B_1}}, \boldsymbol{S_{B_2}}) = (\mathbf{-520}, \mathbf{-450}) + (\mathbf{-330}, \mathbf{-200}) = (\mathbf{-850}, \mathbf{-650}),$$

chooses the strategy B_1.

In the following days, the situation is repeated, the behavior of the choice of strategies by firms *A* and *B* does not change, its results are shown in table 2:

Table 2

n	I	S'_{B_1}	S'_{B_2}	S'_n	j	S'_{A_1}	S'_{A_2}	S'_{A_3}	S'_{A_4}	S''_n	$\bar{S}_n$	r_n	R_n	$\bar{r}_n$
1	3	—190	—250	—250	2	—400	—300	—250	—200	—200	—225	—250	—250	—250
2	4	—520	—450	—260	2	—800	—600	—500	—400	—200	—230	—200	—450	—225
3	4	—850	—650	—283	1	—900	—750	—690	—730	—230	—257	—330	—780	—260
4	3	—1040	—900	—260	1	—1000	—900	—880	—1060	—220	—240	—190	—970	—243
5	3	—1230	—1150	—246	1	—1100	—1050	—1070	—1390	—210	—228	—190	—1160	—232
6	2	—1380	—1450	—242	1	—1200	—1200	—1260	—1720	—200	—221	—150	—1310	—218
7	1	—1480	—1850	—264	2	—1600	—1500	—1510	—1920	—214	—239	—400	—1710	—244
8	2	—1630	—2150	—269	2	—2000	—1800	—1760	—2120	—220	—244	—300	—2010	—251
9	3	—1820	—2400	—267	2	—2400	—2100	—2010	—2320	—223	—245	—250	—2260	—251
10	3	—2010	—2650	—265	2	—2800	—2400	—2260	—2520	—226	—246	—250	—2510	—251
11	3	—2200	—2900	—264	2	—3200	—2700	—2510	—2720	—228	—246	—250	—2760	—251
12	3	—2390	—3150	—263	2	—3600	—3000	—2760	—2920	—230	—246	—250	—3010	—251
13	3	—2580	—3400	—262	2	—4000	—3300	—3010	—3120	—232	—247	—250	—3260	—251
14	3	—2770	—2650	—261	2	—4400	—3600	—3260	—3320	—233	—247	—250	—3510	—251
15	3	—2960	—3900	—260	2	—4800	—3900	—3510	—3520	—234	—247	—250	—3760	—251
16	3	—3150	—4150	—259	2	—5200	—4200	—3760	—3720	—233	—246	—250	—4010	—251
17	4	—3480	—4350	—256	2	—5600	—4500	—4010	—3920	—231	—243	—200	—4210	—248
18	4	—3810	—4550	—253	2	—6000	—4800	—4260	—4120	—229	—241	—200	—4410	—245
19	4	—4140	—4750	—250	2	—6400	—5100	—4510	—4320	—227	—239	—200	—4610	—243
..	..	...	...	...	...	...	...	...	...	...	...	...	...	...

where *n* denote the number of days elapsed after the conclusion of the contract, or a pair of successive strategies («moves») of the firms *A* and *B;*

i denotes the strategy number selected by the company A;

$S'_{B_1}S'_{B_2}$- *denote accumulated (common) losses of the firm A for the first n*

days using the strategies B_1, B_2 of the company *B;*

S'_n- maximum average losses of the firm *A*, which are equal to the maximum accumulated losses for the first *n* days divided by the number of these days;

j – denote the strategy number selected by B.

$S'_{A_1}, S'_{A_2}, S'_{A_3}, S'_{A_4}$are accumulated (general) losses of the firm *A* for the first days according to its strategies respectively A_1, A_2, A_3,A_4 ;

$S'_n$is the minimum average losses of the firm *A*, equal to the minimum accumulated losses for the first *n* days divided by the number of these days;

$\overline{S_n}$denotes an average value of maximum (S'_n) and minimum (S''_n)average losses of firm *A;*

r_n- *denotes real company A losses for each day;*

R_n- *denotes actual accumulated losses of the firm A for n days;*

$\overline{r_n}$is the real average losses of the firm *A* in one day, which are added with the accumulated real losses for the first *n* days divided by the number of these days.

Table 2 shows that with increasing *n* all three values:

S'_n, S''_n i $\overline{S_n}$ approach the exact value of losses (price of the game) of the company *A*, which equals to \$234,2. and were previously found by the graphical method (§1.2), but the average $\overline{S_n}$ coincides relatively faster since $S'_n < -234,2 < S''_n$.

The mixed strategies of the firms *A* and *B* also increase with their exact values as they increase *n* (see §1.2, 1.4), respectively

$\overline{y^*} = (0; 0; 0,685; 0,315), \overline{x^*} = (0,263; 0,737)$, but slowlier.

For example, after *n*=19 repetitions of the game (days), the approximate values of losses of the firm *A*(the price of game) $\overline{S_{19}} = -293\ y.o.$, and the approximate values of mixed strategies of firms *A* i *B*are often determined by their clean strategies:

$$y_{19} = \left(\frac{1}{19}, \frac{2}{19}, \frac{11}{19}, \frac{5}{19}\right) \approx (0,053; 0,105; 0,579; 0,263);$$

$$x_{19} = \left(\frac{4}{19}, \frac{5}{19}\right) \approx (0,211; 0,789).$$

For comparison, the last three columns of table 13 provide real information about the course of the game (each game implementation), which shows that the model (algorithm) adequately reflects the behavior of the players (firms *A* and *B)* during the repetition of the game and allows them to determine their optimal strategies and the price of the game (losses of the company *A).*

It can be seen from the above that the iterative method is practical and universal at the same time. Using it, you can easily find an approximate solution to any matrix game. The volume and complexity of calculations increase relatively slowly as the matrix game size increases.

1.3 Direct Solution of Matrix Games

In principle, any matrix game can be solved by inequalities (15). But it requires a lot of calculations, which increases with the increment of number of players. Therefore, if possible, reduce the number of clearplayer strategies using the «dominance principle» that is as follows[4].

If the elements of some row of the payoff matrix are smaller than the corresponding elements of some other row of the same matrix, then the last row dominate the first. The first row is removed from the matrix. The case with columns is similar, only the column with larger elements is removed.

Further we have to check the inequalities (15). If inequation (15) is fulfilled, then players have pure optimal strategies (the player has the pure maximin strategy and the player the pure minimax). And if not, at least one player's optimal strategies will be mixed.

Let us consider the principle of dominance on the example of the problem of planning the production of by-products (antagonistic case).

1.4 The problem of planning the production of by-products (antagonistic case)

Let it be: in some city there are two enterprises, which in addition to their main products may produce some by-products of the same purpose for the population, but it may be different in design and convenience, etc. Let us suppose that enterprise *A* A_1, A_2, A_3, A_4, A_5, and enterprise *B* produces byproducts of type B_1, B_2, B_3, B_4, B_5. The cost and sales price of all products are the same. Demand forecasting sociologists have determined that N=1000 units will be sold; moreover, if the first enterprise *A* (player I) will

[4] Dyubin G. N., Suzdal V. G. Introduction to Applied Game Theory. Moskow: Science, 1981, 336 p.

produce products of type $\boldsymbol{A_i}$, and the second enterprise *B* (player II) – products of type $\boldsymbol{B_j}$, then the city will find sales $\boldsymbol{p_{ij}N}$ of goods of type$\boldsymbol{A_i}$ and $(\mathbf{1}-\boldsymbol{p_{ij}})\,\boldsymbol{N}$ of goods of type $\boldsymbol{B_j}, \mathbf{0} \le \boldsymbol{p_{ij}} \le \mathbf{1}$.. The capacity of the enterprises is such that each of them can provide the city. Taking the profit from the sale of a unit of goods equal to one, and the usefulness of the player I equals its profit, the payoff matrix *H* of player I can be written as follows:

$$\boldsymbol{H} = (\boldsymbol{p_{ij}\,N})_{\substack{i=1,\dots,m,\\ j=1,\dots,m.}}$$

Similarly, the payoff matrix of player II is written, whose element (i, j) is $(\mathbf{1}-\boldsymbol{p_{ij}})\boldsymbol{N}$. Since in any situation the sum of profits of players I and II is equal to the same number $= \boldsymbol{p_{ij}N} + (\mathbf{1}-\boldsymbol{p_{ij}})\boldsymbol{N}$, an increase of player I winnings is equivalent to a decrease of player II winnings, i.e the interests of players are opposite. Therefore, player II, minimizing sales $\boldsymbol{p_{ij}N}$of goods $\boldsymbol{A_i}$ of player I, maximizes $(\mathbf{1}-\boldsymbol{p_{ij}})\boldsymbol{N}$ sales of his goods B_j Therefore, the game given by the matrix *H,* simulates an antagonistic game.

The solution of the game determines the optimal strategies $\overline{\boldsymbol{x}}^*, \overline{\boldsymbol{y}}^*$for players I and II, respectively, as well as the mathematical expectation of winning of player I is equal to $\boldsymbol{S}(\overline{\boldsymbol{y}}, \overline{\boldsymbol{x}})$ In this game, the mathematical expectation of winning of player II is equal to $-\boldsymbol{S}(\overline{\boldsymbol{y}}, \overline{\boldsymbol{x}})$. Since the sum of goods sold equals to*N*, the mathematical expectation of goods sold by the enterprise *B* equals to $\boldsymbol{N} - \boldsymbol{S}(\overline{\boldsymbol{y}}, \overline{\boldsymbol{x}})$.

Let us consider the solution of the game on a specific numerical example. Suppose that the estimated share of sales of enterprise *A* products is given in Table 3.

Table 3

Підприємство *A*	**Підприємство *B***				
	B_1	**B_2**	**B_3**	**B_4**	**B_5**
A_1	0,5	0,5	0,4	0,5	0,2
A_2	0,5	0,4	0,7	0,1	0,6
A_3	0,2	0,3	0,4	0,1	0,7
A_4	0,3	0,6	0,7	0,3	0,2
A_5	0,4	0,4	0,3	0,0	0,2

It is necessary to determine the types of products produced by each enterprise. In this case, the player's I payoff matrix will look like this

$$\boldsymbol{H} = \begin{pmatrix} 500 & 500 & 400 & 500 & 200 \\ 500 & 400 & 700 & 100 & 600 \\ 200 & 300 & 400 & 100 & 700 \\ 300 & 600 & 700 & 300 & 200 \\ 400 & 400 & 300 & 0 & 200 \end{pmatrix}.$$

Noting that it is enough to solve the game with a matrix of winnings $\boldsymbol{H^1} = \frac{1}{100}\boldsymbol{H},$ i.e

$$\boldsymbol{H^1} = \begin{pmatrix} 5 & 5 & 4 & 5 & 2 \\ 5 & 4 & 7 & 1 & 6 \\ 2 & 3 & 4 & 1 & 7 \\ 3 & 6 & 7 & 3 & 2 \\ 4 & 4 & 3 & 0 & 2 \end{pmatrix}.$$

The game with the payoff matrix H^{1} is called the *subgame* of the game with the matrix *H*. The set of pure strategies of each of the players in the game is contained in the set of its pure strategies in the game itself, from which it follows that the set of mixed strategies of each player in the subgame is contained in the set of the mixed strategies of the game.

We apply the principle of dominance. It is easy to determine that the elements of the fifth row of the matrix H^1 are not greater than the corresponding elements of the first row, and therefore the first strategy of player I dominates the fifth. In addition, the elements of the first and second columns are not less than the corresponding elements of the fourth column. Therefore, player's fourth strategy dominates his first and second strategy. According to the principle of dominance, we remove the fifth row and the first and second columns. Obtain a subgame of the game with the payoff matrix H^{1}, which in the matrix form is given by the matrix

$$\boldsymbol{H^2} = \begin{pmatrix} 4 & 5 & 2 \\ 7 & 1 & 6 \\ 4 & 1 & 7 \\ 7 & 3 & 2 \end{pmatrix}.$$

Note that the *i*th row of the matrix H^2 is corresponded by *i*th strategy, and*j*th column – (*j* + 2)-th strategy of the game $\boldsymbol{H^1}$. Analysis of the matrix $\boldsymbol{H^2}$ shows that the third strategy of player II is dominated by a mixed strategy that uses fourth and fifth strategies with the probabilities 3/5 and 2/5 respectively. According to the principle of dominance, we remove the first column of the matrix $\boldsymbol{H^2}$ and get a subgame with a matrix

$$\boldsymbol{H^3} = \begin{pmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{1} & \mathbf{6} \\ \mathbf{1} & \mathbf{7} \\ \mathbf{3} & \mathbf{2} \end{pmatrix}.$$

any solution of which is the solution of the game H^2, and game $\boldsymbol{H^1}$ i H.

From the analysis of the matrix H^3 it is easy to determine that the elements of the second row are not larger than the corresponding elements of the third row, and the elements of the fourth row are not greater than the corresponding elements of the first row. Therefore, the first and third strategies of player I dominate respectively the fourth and second strategies of player I.

Again, using the dominance principle, we obtain a subgame with a matrix

$$\boldsymbol{H^4} = \begin{pmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{1} & \mathbf{7} \end{pmatrix}.$$

Let us see if the game has a solution in pure strategies, with optimal strategies of players I and II respectively being a pure maximin strategy and a pure minimax strategy. However, if the game with a payoff matrix $\boldsymbol{H^4}$ is not solved in pure strategies, then both players have only optimal strategies that use all their pure strategies with positive probabilities.

The matrix $\boldsymbol{H^4}$ does not have *saddle point,* because the equation of elements is not satisfied

$$\max_i \min_j \boldsymbol{h_{ij}} = \min_j \max_i \boldsymbol{h_{ij}}$$

matrix $\boldsymbol{H^4}$, i.e the optimal strategies of the players are mixed.

Let $\overline{\boldsymbol{x}}$ – be a random mixed strategy of player I. If $\boldsymbol{x_1}$ is the probability of a player's choice of his first strategy in terms of $\overline{\boldsymbol{x}}$, then the probability of him choosing a second strategy is $\mathbf{1} - \boldsymbol{x_1}$. Similarly, if $\overline{\boldsymbol{y}}$ is a random mixed strategy of player II, then it looks like $(\boldsymbol{y_1},\ \mathbf{1} - \boldsymbol{y_1})$. It is easy to prove that the optimal strategies of players I and II

$$\overline{\boldsymbol{x}^*} = (\boldsymbol{x_1^*}, \mathbf{1} - \boldsymbol{x_1^*}), \overline{\boldsymbol{y}^*} = (\boldsymbol{y_1^*}, \mathbf{1} - \boldsymbol{y_1^*})$$

are calculated by the formulas

$$\boldsymbol{x_1^*} = \frac{\boldsymbol{h_{22} - h_{21}}}{\boldsymbol{h_{11} - h_{12} - h_{21} + h_{22}}}, \boldsymbol{y_1^*} = \frac{\boldsymbol{h_{22} - h_{12}}}{\boldsymbol{h_{11} - h_{12} - h_{21} + h_{22}}},$$

and the payment function of the game is equal to

$$\boldsymbol{S}(\overline{\boldsymbol{y}^*}, \overline{\boldsymbol{x}^*}) = \frac{\boldsymbol{h_{11}h_{22} - h_{12}h_{21}}}{\boldsymbol{h_{11} - h_{12} - h_{21} + h_{22}}}.$$

As a result of calculations we get

$$\boldsymbol{x_1^* = 2/3}, \boldsymbol{y_1^* = 5/9}, \boldsymbol{S(y, x) = 11/3}.$$

Strategies $\overline{\boldsymbol{x}^*} = (\boldsymbol{2/3}, \boldsymbol{1/3})$ and $\overline{\boldsymbol{y}^*} = (\boldsymbol{5/9}, \boldsymbol{4/9})$are consistent to strategies $\overline{\boldsymbol{x}^*} = (\boldsymbol{2/3}, \boldsymbol{0}, \boldsymbol{1/3}, \boldsymbol{0}, \boldsymbol{0})$ and $\overline{\boldsymbol{y}^*} = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{5/9}, \boldsymbol{4/9})$ of the initial game. The value of the game with the payoff matrix *H* is equal to 1100/3.

The result means that the enterprise *A* selects the production $\boldsymbol{A_1}$ i $\boldsymbol{A_3}$ with probabilities that are equal to 2/3 and 1/3 respectively, and the enterprise *B* – production B_4 *and* $B_{and\ 5}$ with probabilities of 5/9 and 4/9 respectively. Thus the mathematical expectation of the number of goods sold by enterprises *A* and *B* will be equal to 1100/3 and 1900/3 respectively.

2. Non-zero-sum bi-matrix games

Above, the zero-sum paired games, which are entirely determined by one payment matrix, were considered (Table 12). *The optimal strategies* are the following *strategies* $\overline{\boldsymbol{y}^*}$ *and* $\overline{\boldsymbol{x}^*}$ respectively for the parties *A* and *B*, which satisfy the conditions (15), under which it is not advantageous to deviate from these strategies for any player. This is called the *equilibrium situation.* It proves that zero-sum games always have at least one optimal solution $(\overline{\boldsymbol{y}^*}, \overline{\boldsymbol{x}^*})$, i.e at least one equilibrium point with the price of the game $\boldsymbol{C} = \boldsymbol{S}(\overline{\boldsymbol{y}^*}, \overline{\boldsymbol{x}^*})$. As a rule, such a solution is unique[5].

But, even when there are no such points of equilibrium, the price of the game is always the same and is equal to $\boldsymbol{C} = \boldsymbol{S}(\overline{\boldsymbol{y_i^*}}, \overline{\boldsymbol{x_i^*}})(\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \dots)$. Therefore, such equilibrium points are considered equivalent and in the general case one can assume that zero-sum games always have the only optimal solution.

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[5] Zamkov OO, Tolstenko AV, Cheremnykh Yu.N. Mathematical Methods in Economics. Moskow: DIS, 1997, 368 p.

Unlike zero-sum games, there are non-zero-sum games where it is not necessary for one player to win and the other to lose; they can both win and lose at the same time.

As the interests of players in such games are not completely opposite, their behavior becomes more diverse. For example, if a zero-sum game made it unprofitable for each player to tell his or her strategy to the other (this could reduce his or her winnings), then in a non-zero-sum game, it becomes desirable to coordinate with or influence a partner in some way.

Non-zero-sum games are also called *bimatrix*, as they are defined either by two matrices indicating the payments (winnings) of each party *A* and *B:*

$A \setminus B$	B_1	...	B_n		$A \setminus B$	B_1	...	B_n
A_1	a_{11}	...	a_{1n}		A_1	b_{11}	...	b_{1n}
....	...	...	...		...	...	...	...
A_m	a_{m1}	...	a_{mn}		A_m	b_{m1}	...	b_{mn}

or by one block matrix whose elements are pairs or blocks $(\boldsymbol{a_{ij}}, \boldsymbol{b_{ij}})$,

A/B	B_1	...	B_n
A_1	(a_{11}, b_{11})	...	(a_{1n}, b_{1n})
...	...	...	...
A_m	(a_{m1}, b_{m1})	...	(a_{mn}, b_{mn})

There are two types of bimatrix games – *non-cooperative games,* that prohibit any co-operation of the parties, and *cooperative games,* that allow such cooperation. It is obvious that cooperative games are a more complex object of study (at least because forms of cooperation can be diverse).

3. Non-cooperative games

In most economic, industrial, military, political, environmental, and **adaptive maintenance**administrative-legal conflicts, the purpose of each participant is to obtain as much individual gain as possible. All participants in such conflicts, for example, can win at the same time. Therefore, the non-compliant interests of participants are not quite the opposite, which makes the conflict non-antagonistic. Such a conflict may be modeled *by a non-cooperative game* if it fulfills such conditions.

1. Conflict is determined by the non-antagonistic interaction of the participants.

2. The parties of the conflict cannot (or have no right) to make mutually binding agreements.

3. The parties' own actions are performed independently of each other, that is, each of them has no information about the actions taken by the other party; the results of these actions are estimated by the real numbers that determine the usefulness of the situation for each of the parties.

4. Each of the parties of the conflict knows, both for themselves and for others, the usefulness of any possible situation that may result from their interaction.

3.1 Situations (points) of equilibrium

Let us take a closer look at non-cooperative games. In this case, an important role is played by situations of equilibrium, characterized by the fact that it is disadvantageous for none of the parties to violate them. and earlier, through $\mathbf{y} = (\mathbf{y_1} \ldots, \mathbf{y_m})$, $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_n})$mixed strategies of players A and B.

Then their average winnings will be accordingly equal to

$$\mathbf{S_A}(\bar{\mathbf{y}}, \bar{\mathbf{x}}) = \sum_{\mathbf{i=1}}^{\mathbf{m}} \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{a_{ij}}\, \mathbf{y_i x_j};\ \mathbf{S_B}(\bar{\mathbf{y}}, \bar{\mathbf{x}}) = \sum_{\mathbf{i=1}}^{\mathbf{m}} \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{b_{ij}} \mathbf{y_i x_j}. \quad (12)$$

If among the common strategies there are$\overline{\mathbf{y}^*} = (\mathbf{y_1^*}, \ldots, \mathbf{y_m^*})$ and $\bar{\mathbf{x}}^* = (\mathbf{x_1^*}, \ldots, \mathbf{x_n^*})$that satisfy the conditions

$$S_A(\bar{y}, \overline{x^*}) \le S_A(\overline{y^*}, \overline{x^*});\ S_B(\overline{y^*}, \bar{x}) \le S_B(\overline{y^*}, \overline{x^*}) \quad (13)$$

then using $\overline{\overline{\mathbf{y}^*}}^*$ and $\bar{\mathbf{x}}^*$ creates an equilibrium situation.

The theory holds that every non-cooperative bimatrix game has at least one equilibrium situation (point) determined by inequations (13). When such a point (pair) $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{x}})$ is unique, it can be considered as the optimal strategies $\overline{\boldsymbol{y}}^*$and $\overline{\boldsymbol{x}}^*$ of the sides *A* and *B.*

Uncertainty arises when there is more than one equilibrium point that satisfies conditions (27). And, unlike zero-sum games, the winnings of the parties *A* and *B* in these points differ – they are not equivalent.

Consider this situation using a simple example.

Let the block payment matrix (Table 4) look like this

Table 4

A \ B	B_1	B_2
A_1	(7,3)	(0,0)
A_2	(0,0)	(3,7)

By a straightforward substitution of formula (12), it is easy to check that pure strategies are $\overline{y_1}^* = (\mathbf{1},\mathbf{0})$, $\overline{x_1}^* = (\mathbf{1},\mathbf{0})$ and

$\overline{y_2}^* = (\mathbf{1},\mathbf{0}), \overline{x_2}^* = (\mathbf{0},\mathbf{1})$satisfy the equilibrium conditions. The winnings of the parties *A and B* at these points of equilibrium are respectively equal to

$$S_A(\overline{y_1}^*, \overline{x_1}^*) = 7;\ S_B(\overline{y_1}^*, \overline{x_1}^*) = 3;$$

$$S_A(\overline{y_2}^*, \overline{x_2}^*) = 3;\ S_B(\overline{y_2}^*, \overline{x_2}^*) = 7;$$

Now let us check whether there are points of equilibrium among the mixed strategies of the parties *A* and *B.*

Since

$$y_1 + y_2 = 1; x_1 + x_2 = 1; m = n = 2,$$

then from relations (13) and Table 15 it implies that the average winnings of the parties *A* and *B* are respectively equal to

$$S_A(\bar{y}, \bar{x}) = 7y_1x_1 + 3y_2x_2 = 7y_1x_1 + 3(1 - y_1)(1 - x_1), \quad (14)$$

$$S_B(\bar{y}, \bar{x}) = 3y_1x_1 + 7y_2x_2 = 3y_1x_1 + 7(1 - y_1)(1 - x_1),$$

that is, S_A and S_B are functions from two variables y_1 and x_1:

$$S_A(y_1, x_1) = 7y_1x_1 + 3(1 - y_1)(1 - x_1);$$

$$S_B(y_1, x_1) = 3y_1x_1 + 7(1 - y_1)(1 - x_1).$$

The equilibrium situation is characterized by the fact that it is not profitable for the side *A* to change its strategy y_1, and for the side *B* – its strategy x_1, because this will reduce their average winnings. It follows that the equilibrium conditions in this case have the form

$$\begin{cases} \dfrac{\delta S_A}{\delta y_1} = 7x_1 - 3(1 - x_1) = 0, \\ \dfrac{\delta S_B}{\delta x_1} = 3y_1 - 7(1 - y_1) = 0. \end{cases}$$

Solving this system of equations, we find the third equilibrium point among the mixed strategies for the sides *A* and *B:*

$$\boldsymbol{y_1}^* = \mathbf{0,7}; \boldsymbol{y_2}^* = \mathbf{1 - 0,7 = 0,3}; \boldsymbol{x_1}^* = \mathbf{0,3};\ \boldsymbol{x_2}^* = \mathbf{1 - 0,3 = 0,7},$$

that is

$$\overline{\boldsymbol{y}}_3{}^* = (\mathbf{0,7; 0,3});\ \overline{\boldsymbol{x}}_3{}^* = (\mathbf{0,3; 0,7})$$

with the winnings calculated by the formulas (28):

$$\boldsymbol{S_A}(\overline{\boldsymbol{y}}_3{}^*, \overline{\boldsymbol{x}}_3{}^*) = \mathbf{2,1};\ \boldsymbol{S_B}(\overline{\boldsymbol{y}}_3{}^*, \overline{\boldsymbol{x}}_3{}^*) = \mathbf{2,1}.$$

It is easy to check that the equilibrium conditions (27) are satisfied at this point:

$$\boldsymbol{S_A}(\overline{\boldsymbol{y}}, \overline{\boldsymbol{x}}_3{}^*) = \mathbf{7}\boldsymbol{y_1} \cdot \mathbf{0,3} + \mathbf{3}(\mathbf{1} - \boldsymbol{y_1}) \cdot \mathbf{0,7} = \mathbf{2,1} = \boldsymbol{S_A}(\overline{\boldsymbol{y}}_3{}^*, \overline{\boldsymbol{x}}_3{}^*),$$

$$\boldsymbol{S_B}(\overline{\boldsymbol{y}}_3{}^*, \overline{\boldsymbol{x}}) = \mathbf{3}\boldsymbol{x_1} \cdot \mathbf{0,7} + \mathbf{7}(\mathbf{1} - \boldsymbol{x_1}) \cdot \mathbf{0,3} = \mathbf{2,1} = \boldsymbol{S_B}(\overline{\boldsymbol{y}}_3{}^*, \overline{\boldsymbol{x}}_3{}^*).$$

Obviously, the first situation (point) of equilibrium is more favorable for the side *A*, the second – for the side *B.* In the third equilibrium point, the parties' gains are the same, but they are smaller than in the first and second points. In the end, it is difficult to understand what the outcome of the parties *A* та *B* may be and how they should behave.

Thus, if there is more than one point (situation) of equilibrium, unambiguous recommendations for the choice of optimal strategies for the parties *A* and *B* cannot be given. In many cases, mutual contacts and agreements between the parties *A* and *B* make it possible.

In general, non-cooperative games are examined on a case-by-case basis.

3.2 The problem of planning the production of the by-product (non-antagonistic case)

Let us consider the problem of planning the production of the by-product (non-antagonistic case).

Suppose that two enterprises can produce by-products in the same production conditions as in the antagonistic case, but the possibility of selling these products has changed.

Now, according to sociologists, if the first enterprise (player I) will produce products of type $\boldsymbol{A_i}(\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{m})$, and the second (player II) – products of type $\boldsymbol{B_j}(\mathbf{1} \le \boldsymbol{j} \le \boldsymbol{n})$, then the city will find sales $a_{ij}$of goods of type $\boldsymbol{A_i}$ and sales $\boldsymbol{b_{ij}}$ of goods of type $\boldsymbol{B_j}$**.**

Since the sale of products of any enterprise depends on what products the other enterprise produces, and each enterprise tries to maximize the volume of sales, we have a production-trade conflict. This conflict is modeled by the game of the same players I and II with the same respectively *m* and *n* strategies as in the antagonistic game.

But this game is non-antagonistic, since the amount of products sold will now depend on the situation.

Taking the profit from the sale of units of goods equal to one, and the utility of players I and II equal their income, we model this conflict by a bi-matrix game given by a pair of matrices

$$\boldsymbol{A} = (\boldsymbol{a_{ij}})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \text{ i } \boldsymbol{B} = (\boldsymbol{b_{ij}})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$$

where $\boldsymbol{a_{ij}}$and $\boldsymbol{b_{ij}}$ – wins of the players I and II respectively in the situation*(i, j)*.

Consider the solution of this game on a specific numerical example, assuming that companies I and II plan to produce by-products of types $\boldsymbol{A_i}\ (\boldsymbol{i = 1, 2})$ and$\boldsymbol{B_j}\ (\boldsymbol{j = 1, 2})$, respectively, and the expected profits from the sale of these products are given by the matrices:

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{600} & \mathbf{300} \\ \mathbf{300} & \mathbf{900} \end{pmatrix} \text{ i } \boldsymbol{B} = \begin{pmatrix} \mathbf{500} & \mathbf{1500} \\ \mathbf{2000} & \mathbf{500} \end{pmatrix}.$$

It is necessary to determine the type of products that make sense for each enterprise.

Let us denote

$$\boldsymbol{A^*} = \boldsymbol{a_{11}} - \boldsymbol{a_{12}} - \boldsymbol{a_{21}} + \boldsymbol{a_{22}}, \boldsymbol{a} = \boldsymbol{a_{22}} - \boldsymbol{a_{12}};$$

$$\boldsymbol{B^*} = \boldsymbol{b_{11}} - \boldsymbol{b_{12}} - \boldsymbol{b_{21}} + \boldsymbol{b_{22}};, \boldsymbol{b} = \boldsymbol{b_{22}} - \boldsymbol{b_{21}}.$$

If$\boldsymbol{A^*} \neq \mathbf{0}$ and$\boldsymbol{B^*} \neq \mathbf{0}$, then the game has a balance of mixed strategies, namely

$$\overline{\boldsymbol{x}}^* = (\boldsymbol{x_1^*}, \mathbf{1} - \boldsymbol{x_1^*}), \overline{\boldsymbol{y}}^* = (\boldsymbol{y_1^*}, \mathbf{1} - \boldsymbol{y_1^*})$$

where

$$\boldsymbol{x_1^*} = \frac{\boldsymbol{b}}{\boldsymbol{B^*}}, \boldsymbol{y_1^*} = \frac{\boldsymbol{a}}{\boldsymbol{A^*}}.$$

As a result of calculations we get

$$\boldsymbol{A^*} = \mathbf{900}, \boldsymbol{a} = \mathbf{600};\ \boldsymbol{B^*} = \mathbf{-2500}, \boldsymbol{b} = \mathbf{-1500}.$$

Therefore, the equilibrium situation is formed by vectors

$$\bar{\boldsymbol{x}}^* = (\mathbf{3/5}\,, \mathbf{2/5}), \bar{\boldsymbol{y}}^* = (\mathbf{2/3}\,, \mathbf{1/3})$$

and the mathematical expectation of the winnings of players I and II in the equilibrium situation will accordingly be

$$\boldsymbol{S_A}(\bar{\boldsymbol{y}}^*, \bar{\boldsymbol{x}}^*) = (\boldsymbol{a_{11}} - \boldsymbol{a_{12}} - \boldsymbol{a_{21}} + \boldsymbol{a_{22}})\boldsymbol{x_1^*}\boldsymbol{y_1^*} + (\boldsymbol{a_{12}} - \boldsymbol{a_{22}})\boldsymbol{x_1^*} +$$

$$+(\boldsymbol{a_{21}} - \boldsymbol{a_{22}})\boldsymbol{y_1^*} + \boldsymbol{a_{22}} = \mathbf{500}$$

$$\boldsymbol{S_B}(\bar{\boldsymbol{y}}^*, \bar{\boldsymbol{x}}^*) = (\boldsymbol{b_{11}} - \boldsymbol{b_{12}} - \boldsymbol{b_{21}} + \boldsymbol{b_{22}})\boldsymbol{x_1^*}\boldsymbol{y_1^*} + (\boldsymbol{b_{12}} - \boldsymbol{b_{22}})\boldsymbol{x_1^*} +$$

$$+(\boldsymbol{b_{21}} - \boldsymbol{b_{22}})\boldsymbol{y_1^*} + \boldsymbol{b_{22}} = \mathbf{1100}.$$

The result means that the enterprise *A* selects the production of type A_1 and A_2 with probabilities that are equal to 3/5 and 2/5 respectively, and the enterprise *B* – production of type B_1 and B_2 with probabilities of 2/3 and 1/3 respectively. Thus the mathematical expectation of the number of goods sold by enterprises *A* and *B* will be equal to \$500 and \$1100 respectively.

4. Cooperative games

4.1 Problem Statement

Most non-antagonistic conflicts in the economy and related industries are characterized by the fact that their participants can join forces through cooperation. Cooperation between players results in a qualitatively new conflict compared to a non-cooperative case.

As we have seen, in non-cooperative games, deviating one of the participants from the equilibrium situation does not give him any advantage. But if several players deviate, they can earn more than in the equilibrium situation. Therefore, in conditions where cooperation between players is possible, the principle of equilibrium does not come true.

For example, let a non-antagonistic game be given by the following matrices:

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{10} & \mathbf{1} \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} \mathbf{5} & \mathbf{10} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Here, the only equilibrium situation will be a situation (0,0) in which each player chooses his or her second pure strategy and wins a unit.

However, it is obvious that if players agree and choose their first pure strategies, then in the situation (1,1), each of them will win five units.

However, it is clear that this situation, which may arise in the case of cooperation, is rather unstable, since each player, randomly changing his strategy, increases his winnings.

4.2 By-Product Production Planning Problem (Cooperative Case)

Let two enterprises produce by-products under production conditions adopted as in antagonistic case, but taking into account sales opportunities, as in a non-cooperative case. Then, as it was established, such a conflict is modeled by a finite game of two persons with a non-zero sum given by a pair of $\boldsymbol{m} \times \boldsymbol{n}$ matrices $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ and $\boldsymbol{B} = (\boldsymbol{b_{ij}})$ elements of which are the winnings (in units of utility) of players I and II respectively, if they are chosen respectively by their *i*-th and *j*-th pure strategies.

Now, in this game, given the nature of the conflict, it is allowed to cooperate without transferring utility from one player to another, that is, players can make agreements and choose a compatible strategy$\bar{\boldsymbol{z}}$.

Obviously,

$$\bar{z} = (z_{11}, \dots, z_{ij}, \dots, z_{mn}) z_{ij} \geq 0, \sum_{i,j} z_{ij} = 1,$$

where $\boldsymbol{z_{ij}}$ – denotes the probability of choosing respectively compatible strategies (i, j) by players I and II.

The mathematical expectation of winning, respectively, players I and II under the conditions of their strategy is naturally determined by the formulas

$$S_A(\bar{z}) = \sum_{i,j} a_{ij}, z_{ij},$$

$$S_B(\bar{z}) = \sum_{i,j} b_{ij}, z_{ij}.$$

The points $(\boldsymbol{S_A}(\bar{\boldsymbol{z}}), \boldsymbol{S_B}(\bar{\boldsymbol{z}}))$ form *the valid set R.*

By agreement, players can get as a win a random vector of this set $(\bar{\boldsymbol{S}}_{\boldsymbol{A}}(\bar{\boldsymbol{z}}), \bar{\boldsymbol{S}}_{\boldsymbol{B}}(\bar{\boldsymbol{z}}))$.

Obviously, with compatible actions, players I and II must win no less than the values as if playing the antagonistic game $\boldsymbol{S_A}(\bar{\boldsymbol{y}}^*, \bar{\boldsymbol{x}}^*)$ and $\boldsymbol{S_B}(\bar{\boldsymbol{y}}^*, \bar{\boldsymbol{x}}^*)$, calculated by formula (26), which are players' winnings when they fail to reach an agreement.

To find $\left(\bar{S}_A(\bar{z}), \bar{S}_B(\bar{z})\right)$ use the following*arbitration scheme.*

1. The beginning of coordinates is transferred to a point with coordinates $S_A(\bar{y}^*, \bar{x}^*)$ and $S_B(\bar{y}^*, \bar{x}^*)$, that is, this point is transferred to a point (0,0), where the set P becomes the setP'.

2.There is a single point with the coordinates $\bar{S'}_A(\bar{y}^*, \bar{x}^*)$ and $\bar{S'}_B(\bar{y}^*, \bar{x}^*)$ with P' where $\bar{S'}_A(\bar{y}^*, \bar{x}^*) > 0$ and $\bar{S'}_B(\bar{y}^*, \bar{x}^*) > 0$, and

$\bar{S'}_A(\bar{y}^*, \bar{x}^*)\bar{S'}_B(\bar{y}^*, \bar{x}^*)$is the maximum of all earnings

$$S'_A(\bar{y}^*, \bar{x}^*)S'_B(\bar{y}^*, \bar{x}^*).$$

3. We find the arbitration solution by inverse transformation of utility relative to $\bar{S'}_A(\bar{y}^*, \bar{x}^*)$ and $\bar{S'}_B(\bar{y}^*, \bar{x}^*)$.

Let us find an arbitration solution for specific data of the problem of planning the production of by-products in a non-cooperative case, that is, let a cooperative game without side payments be given by the following matrices:

$$A = \begin{pmatrix} 600 & 300 \\ 300 & 900 \end{pmatrix} \text{ and } B = \begin{pmatrix} 500 & 1500 \\ 2000 & 500 \end{pmatrix}.$$

In the non-cooperative case, the equilibrium vectors were vectors $\bar{\boldsymbol{x}}^*$= (3/5, 2/5), $\bar{\boldsymbol{y}}^*$ = (2/3, 1/3). As it has been explored, in a non-cooperative bimatrix game, where cooperation is neglected and players choose their strategies independently, the mathematical expectation of winning of the player I is equal to $\boldsymbol{S_A}(\bar{\boldsymbol{y}}^*, \bar{\boldsymbol{x}}^*) = \mathbf{500}$ and player II -$\boldsymbol{S_B}(\bar{\boldsymbol{y}}^*, \bar{\boldsymbol{x}}^*) = \mathbf{1100}$.

Now suppose that players can cooperate and choose a compatible mixed strategy without passing on utility to one another.

We transform the coordinates by moving the origin to the point (500, 1100) by the formulas

$$S_A(\bar{y}^*, \bar{x}^*) = S_A(\bar{z}) - 500,$$

$$S_B(\bar{y}^*, \bar{x}^*) = S_B(\bar{z}) - 1100,$$

thus constructing the areaP'.

Let us find the point with the coordinates $\bar{S'}_A(\bar{y}^*, \bar{x}^*)$ and $\bar{S'}_B(\bar{y}^*, \bar{x}^*)$that maximizes the function

$$S'_B = \overline{S'}_A(\bar{y}^*, \bar{x}^*)\overline{S'}_B(\bar{y}^*, \bar{x}^*)$$

на множині P' при $\overline{S'}_A(\bar{y}^*, \bar{x}^*) > 0$ і $\overline{S'}_B(\bar{y}^*, \bar{x}^*) > 0$.

The equation of the line passing through the points (-200, 900) and (400, -600) has the form

$$\boldsymbol{S'_B(\bar{y}^*, \bar{x}^*) = -\frac{5}{2}S'_A(\bar{y}^*, \bar{x}^*) + 400}.$$

Substituting this into function $\boldsymbol{S'_{AB}}$, we differentiate the result expression, equate the derivative to zero, solve the obtained equation with respect to $\boldsymbol{\overline{S'}_A(\bar{y}^*, \bar{x}^*)}$, and find

$$\boldsymbol{\overline{S'}_A(\bar{y}^*, \bar{x}^*) = 80}, \text{a } \boldsymbol{\overline{S'}_B(\bar{y}^*, \bar{x}^*) = 200}.$$

Next, by inverse transformation, we find the arbitration solution for the original cooperative game:

$$\boldsymbol{(\bar{S}_A(\bar{z}) = 580, \bar{S}_B(\bar{z}) = 1300)}.$$

The arbitration award can be implemented by applying a compatible mixed strategy $\boldsymbol{\bar{z} = (0, 0, z_{21}, z_{22})}$ The strategy j components are found from the formulas for calculations $\boldsymbol{\bar{S}_A(\bar{z}), \bar{S}_B(\bar{z})}$, substituting $\boldsymbol{S_A(\bar{z}) = \bar{S}_A(\bar{z}), S_B(\bar{z}) = \bar{S}_B(\bar{z})}$.

In particular, we find $\boldsymbol{z_{21} = 8/15}$, $\boldsymbol{z_{22} = 7/15}$, according to which player I uses only the second strategy, and player II applies the first and second accordingly with probabilities 8/15 and 7/15. In this case, the agreement between the players leads to the fact that the mathematical expectation of winning players I and II will accordingly equal $580. ($500 in non-cooperative case) and $1300 ($1100 in the non-cooperative case).

Thus, cooperating in a non-antagonistic conflict increases the mathematical expectation of winning (in the sense of utility) of each player.

5. Optimizing product quality control

Let us consider, for example, using an example of the optimization of product quality control, the non-cooperative case and the case of players' cooperation[6].

[6] Ivanilov Yu. P., Lotov A. V. Mathematical models in economics, Moskow.: Science, 1979, 304 p.

5.1 Problem statement

Let some products, manufactured by the supplier company *A* (raw materials for light industry, primary agricultural production, etc.), be supplied to the enterprise *B* for the recycling and manufacturing of finished products (clothing, shoes, food, etc.). Each enterprise is interested in increasing its profits. In this regard, the enterprise *B* controls the quality of the products of the enterprise *A*, and the enterprise *A* is not always interested in improving its quality.

As the control frequency decreases, impunity for product suppliers increases, which in pursuit of quantitative indicators weaken attention to product quality.

As the control frequency increases, the quality of the products of company *B* improves, but the cost of control increases. It is necessary to determine the optimal frequency of control over the quality of products of the enterprise *A* by enterprise *B*,, as well as the optimal enterprise *A* strategy to increase their profits.

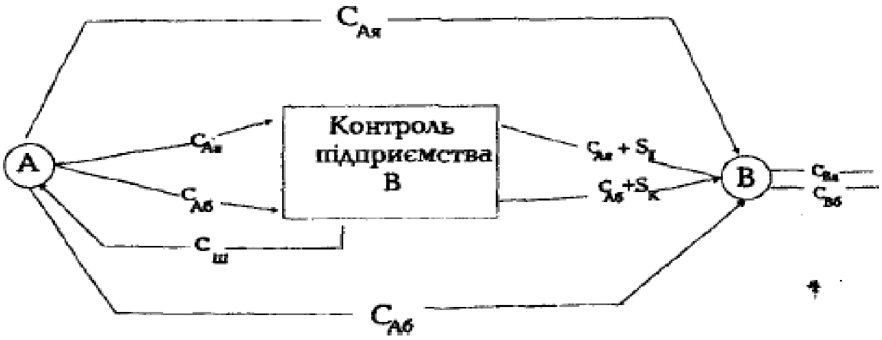

Fig. 1

Let us enter the symbols:

$C_{Ая}$ i $S_{Ая}$- respectively the price and cost of quality products of the enterprise *A*;

$\boldsymbol{C}_{\boldsymbol{A}б}$ i $\boldsymbol{S}_{\boldsymbol{A}б}$ – the corresponding price and cost of the defective products of the enterprise *A;*

$\boldsymbol{C}_{Вб}$ i $C_{Вя}$ ~ respectively the prices of defective and quality products of the enterprise *B;*

$\boldsymbol{S}_{В}$ – cost of manufacturing of products by the enterprise *B;*

$\boldsymbol{S}_{к}$- cost of control for the enterprise *B*;

$C_{ш}$ – the cost of the fine paid to the State bythe enterprise *A* in the case of finding a defect.

We present graphically the movement of products from the enterprise *A* to the enterprise *B* (Fig. 4).

5.2 Non-cooperative case

We use the theory of non-cooperative games to solve this problem. Let us denote by $\boldsymbol{y}_{я}$ the probability of producing quality products by the company *A* (strategy$A_{\mathbf{1}}$), and by $\boldsymbol{y}_{б}$ – defective ones (strategy A_2), while$\boldsymbol{y}_{я}+\boldsymbol{y}_{б}=\mathbf{1}$. Let us denote by $x_к$ the probability of production control of the enterprise *B* (strategy $\boldsymbol{B}_{\mathbf{1}}$), and by $x_в$ – the probability of lack of control (strategy B_2), $x_к + x_в = 1$. Let us draw up the matrix of wins (profits) for enterprises *A* and *B* respectively (Tables 5 and 6).

Table 5

A\B	$x_к$	$x_в$
$y_я$	$C_{Aя}–S_{Aя}$	$C_{Aя}–S_{Aя}$
$y_б$	$C_{Aб}–S_{Aб}–C_{ш}$	$C_{Aя}–S_{Aб}$

Table 6

A\B	$x_к$	$x_в$
$y_я$	$C_{Bя}–C_{Aя}–S_к–S_B$	$C_{Bя}–C_{Aя}–S_B$
$y_б$	$C_{Bб}–C_{Aб}–S_к–S_B$	$C_{Bб}–C_{Aя}–S_B$

Then their average profits (winnings) according to formulas (26) will be equal to

$$S_A=(C_{Aя}-S_{Aя})y_я x_к+(C_{Aя}-S_{Aя})y_я x_в+$$
$$+(C_{Aб}-S_{Aб}-C_{Ш})y_б x_к+(C_{Aя}-S_{Aб})y_б x_в;$$

$$S_B=(C_{Bя}-C_{Aя}-S_к-S_в)y_я x_к+(C_{Bя}-C_{Aя}-S_в)y_я x_в+$$
$$+(C_{Bб}-C_{Aб}-S_к-S_в)y_я x_в+(C_{Bб}-C_{Aя}-S_в)y_б x_в.$$

Using the notation

$$\boldsymbol{y}_я=\boldsymbol{y},\boldsymbol{y}_б=\mathbf{1}-\boldsymbol{y},\boldsymbol{x}_к=\boldsymbol{x},\boldsymbol{x}_в=\mathbf{1}-\boldsymbol{x},$$

we get

$$\boldsymbol{S}_A=\boldsymbol{y}(\boldsymbol{C}_{Aя}-\boldsymbol{S}_{Aя})+(\mathbf{1}-\boldsymbol{y})[\boldsymbol{x}(\boldsymbol{C}_{Aб}-\boldsymbol{C}_ш-\boldsymbol{C}_{Aя})+\boldsymbol{C}_{Aя}-\boldsymbol{S}_{Aб}]; \quad (15)$$

$$\boldsymbol{S}_B=\boldsymbol{y}(\boldsymbol{C}_{Bя}-\boldsymbol{C}_{Aя}-\boldsymbol{S}_B-\boldsymbol{S}_к\boldsymbol{x})+$$
$$+(\mathbf{1}-\boldsymbol{y})[\boldsymbol{x}(\boldsymbol{C}_{Aя}-\boldsymbol{C}_{Aб}-\boldsymbol{S}_к)+\boldsymbol{C}_{Bб}-\boldsymbol{C}_{Aя}-\boldsymbol{S}_B].$$

The equilibrium situation in this problem is characterized by such an optimal pair (point) (y*, x*) – the optimal frequency (probability) of control x* of the enterprise *A* by the enterprise *B* and the optimal frequency (probability)y^* of production of quality products by the enterprise *A*,, in which it is unprofitable for the side *B* to change its strategy x*, and for the side *A* to change its strategy *y**, as it will decrease the average profits (winnings). The equilibrium conditions are:

$$\frac{\delta S_{\text{А}}}{\delta y} = C_{\text{Ая}} - S_{\text{Ая}} - [x(C_{\text{Аб}} - C_{\text{ш}} - C_{\text{Ая}}) + C_{\text{Ая}} - S_{\text{Аб}}] = 0;$$

$$\frac{\delta S_{\text{В}}}{\delta x} = -yS_{\text{к}} + (1-x)(C_{\text{Ая}} - C_{\text{Аб}} - S_{\text{к}}) = 0.$$

Solving this system of equations we obtain

$$y^* = y^*_{\text{я}} = \frac{C_{\text{Ая}} - C_{\text{Аб}} - S_{\text{к}}}{C_{\text{Ая}} - C_{\text{Аб}}};$$

$$y^*_{\text{б}} = 1 - y^*_{\text{я}} = \frac{S_{\text{к}}}{C_{\text{Ая}} - C_{\text{Аб}}};$$

$$x^* = x^*_{\text{к}} = \frac{S_{\text{Ая}} - S_{\text{Аб}}}{C_{\text{Ая}} - C_{\text{Аб}} + C_{\text{ш}}}; \qquad (16)$$

$$x^*_{\text{в}} = 1 - x^*_{\text{к}} = 1 - \frac{S_{\text{Ая}} - S_{\text{Аб}}}{C_{\text{Ая}} - C_{\text{Аб}} + C_{\text{ш}}}$$

It follows that for any non-zero control value $\boldsymbol{S}_{\text{к}}$ for enterprise *B* there is some optimum defective part for the enterprise *A*, which is equal to $\boldsymbol{y}^*_{\text{б}}$. In order to reduce the critical control frequency $\boldsymbol{x}^*_{\text{к}}$ of the enterprise *B*, it is necessary to increase the value of the fine $\boldsymbol{C}_{\text{ш}}$.

Substituting the obtained values $\boldsymbol{y}^*$ and $\boldsymbol{x}^*$, calculated by the formulas (30), into the relation (29), we obtain the expected optimal profits (wins) of the enterprise *A* and *B*.

$$\boldsymbol{S}^*_{\boldsymbol{A}} = \frac{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}} - \boldsymbol{S}_{\text{к}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}}(\boldsymbol{C}_{\text{Ая}} - \boldsymbol{S}_{\text{Ая}}) + \frac{\boldsymbol{S}_{\text{к}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}} \cdot$$
$$\cdot \left[\frac{\boldsymbol{S}_{\text{Ая}} - \boldsymbol{S}_{\text{Аб}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}} + \boldsymbol{C}_{\text{ш}}}(\boldsymbol{C}_{\text{Аб}} - \boldsymbol{C}_{\text{ш}} - \boldsymbol{C}_{\text{Ая}}) + \boldsymbol{C}_{\text{Ая}} - \boldsymbol{S}_{\text{Аб}}\right];$$

$$\boldsymbol{S}^*_{\boldsymbol{B}} = \frac{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}} - \boldsymbol{S}_{\text{к}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}}\begin{pmatrix} \boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Ая}} - \boldsymbol{S}_{\text{в}} - \\ -\boldsymbol{S}_{\text{к}}\dfrac{\boldsymbol{S}_{\text{Ая}} - \boldsymbol{S}_{\text{Аб}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}} + \boldsymbol{C}_{\text{ш}}} \end{pmatrix} + \frac{\boldsymbol{S}_{\text{к}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}} \cdot$$
$$\cdot \left[\frac{\boldsymbol{S}_{\text{Ая}} - \boldsymbol{S}_{\text{Аб}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}} + \boldsymbol{C}_{\text{ш}}}(\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}} - \boldsymbol{S}_{\text{к}}) + \boldsymbol{C}_{\text{Вб}} - \boldsymbol{C}_{\text{Ая}} - \boldsymbol{S}_{\text{к}}\right];$$

after simplification we have

$$\boldsymbol{S}_{\boldsymbol{A}}^{*} = \boldsymbol{C}_{\text{Ая}} - \boldsymbol{S}_{\text{Ая}};\ \boldsymbol{S}_{\boldsymbol{B}}^{*} = \boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Ая}} - \boldsymbol{S}_{\text{в}} - \frac{\boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Вб}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}}\boldsymbol{S}_{\text{к}}. \quad (17)$$

5.3 Cooperative Case

The theory of non-cooperative games was used above to solve the problem, that is, the situation was considered when the enterprises *A* i *B* did not have any agreements (cooperation) about increasing own profits – each company operates at its own discretion. In this case, the total profit at their optimal strategies is equal to

$$S_A{}^{*} + S_B{}^{*} = C_{\text{Вя}} - S_{\text{Ая}} - S_{\text{в}} - \frac{C_{\text{Вя}} - C_{\text{Вб}}}{C_{\text{Ая}} - C_{\text{Аб}}} S_{\text{к}}.$$

Now let us suppose that between the enterprises *A* and *B* there is an agreement to join their efforts in order to increase the total profit. In particular, this may be the case when an enterprise *B* absorbs an enterprise *A*. In this case, they have one goal – to increase the total profit – which corresponds with one payoff matrix (profit) equal to the sum of the payoff matrices separately for enterprises *A* and *B* (tables 5 and 6):

Table 7

$A \backslash B$	$x_{\text{к}}$	$x_{\text{в}}$
$y_{\text{я}}$	$C_{\text{Вя}}–S_{\text{Ая}}–S_{\text{к}}–S_{\text{В}}$	$C_{\text{Вя}}–S_{\text{Ая}}–S_{\text{В}}$
$y_{\text{б}}$	$C_{\text{Вб}}–S_{\text{Аб}}–S_{\text{к}}–S_{\text{В}}–C_{\text{ш}}$	$C_{\text{Вб}}–S_{\text{Аб}}–S_{\text{В}}$

Since the elements of the second column of this matrix (Table 7) are larger than the corresponding elements of the first column, then for arbitrary strategies of the enterprise *A* the second strategy of the enterprise *B*,, which is characterized by the lack of control over the products of the enterprise *A* ($x_{\text{к}}$=0; $x_{\text{в}}$=1), is optimal for increasing the overall profit of the enterprises *A* and *B,* which average (expected) value in this case is

$$S^{*}{}_{A+B} = (C_{\text{Вя}} - S_{\text{Ая}} - S_{\text{В}})y_{\text{я}} + (C_{\text{Вб}} - S_{\text{Аб}} - S_{\text{В}})y_{\text{б}} =$$
$$= (C_{\text{Вя}} - S_{\text{Ая}} - S_{\text{В}})(1 - y_{\text{б}}) + (C_{\text{Вб}} - S_{\text{Аб}} - S_{\text{В}})y_{\text{б}}. \quad (18)$$

Due to the fact that the profit from the sale of quality products is higher than from the defective ones,

$$\boldsymbol{C}_{\text{Вя}} - \boldsymbol{S}_{\text{Ая}} - \boldsymbol{S}_{\text{В}} > \boldsymbol{C}_{\text{Вб}} - \boldsymbol{S}_{\text{Аб}} - \boldsymbol{S}_{\text{В}},$$

and unlike the first case, when an enterprise *A* works only for its own profit and it is profitable for it to produce some defective products y_6*, in order to increase the total profit 5 $*_{\text{A+B}}$ it wants (is interested) to reduce this proportion. When$_{\text{b}}$= 0, the total profit equals to

$$S^*{}_{A+B} = C_{\text{Вя}} - S_{\text{Ая}} - S_{\text{В}}.$$

We calculate how much greater the total profit of enterprises *A* and *B* are, when they work together, from the total profit when they work separately, each for its own result (see (31), (32)):

$$\Delta_{\boldsymbol{AB}} = \boldsymbol{S}^*{}_{\boldsymbol{A+B}} - (\boldsymbol{S}^*{}_{\boldsymbol{A}} + \boldsymbol{S}^*{}_{\boldsymbol{B}}) = (\boldsymbol{C}_{\text{Вя}} - \boldsymbol{S}_{\text{Ая}} - \boldsymbol{S}_{\text{В}})(\boldsymbol{1} - \boldsymbol{y}_{\text{б}}) +$$
$$+(\boldsymbol{C}_{\text{Вб}} - \boldsymbol{S}_{\text{Аб}} - \boldsymbol{S}_{\text{В}})\boldsymbol{y}_{\text{б}} - \boldsymbol{C}_{\text{Вя}} + \boldsymbol{S}_{\text{Ая}} + \boldsymbol{S}_{\text{В}} + \frac{\boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Вб}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}}\boldsymbol{S}_{\text{к}};$$

after simplifications

$$\Delta_{\boldsymbol{AB}} = -(\boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Вб}} - \boldsymbol{S}_{\text{Ая}} + \boldsymbol{S}_{\text{Аб}})\boldsymbol{y}_{\text{б}} + \frac{\boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Вб}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}}\boldsymbol{S}_{\text{к}}. \qquad (19)$$

Since the value of the expression in parentheses is always positive, the difference $\Delta_{\boldsymbol{AB}}$ is a linear descending function relative to $\boldsymbol{y}_{\text{б}}$(the share of defective products of the enterprise *A).* Therefore, the maximum difference value looks like

$$\Delta_{\boldsymbol{AB}}^{(\boldsymbol{max})} = \frac{\boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Вб}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}}\boldsymbol{S}_{\text{к}}.$$

when $\boldsymbol{y}_{\text{б}} = \boldsymbol{0}$

Let enterprise A, working with company *B*, produce the same proportion of defective products $y_{б}$* when it works independently.

Substituting $\boldsymbol{y}_{\text{б}}{}^*$, which is determined by relations (16), into expression (33), we obtain

$$\Delta^*{}_{\text{AB}} = -(\boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Вб}} - \boldsymbol{S}_{\text{Ая}} + \boldsymbol{S}_{\text{Аб}})\frac{\boldsymbol{S}_{\text{к}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}} + \frac{\boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Вб}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}}\boldsymbol{S}_{\text{к}}.$$

after simplifications

$$\Delta^*{}_{\text{AB}} = \frac{\boldsymbol{S}_{\text{Ая}} - \boldsymbol{S}_{\text{Аб}}}{\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}}}\boldsymbol{S}_{\text{к}}$$

which is obviously less than $\Delta_{\boldsymbol{AB}}{}^{(\boldsymbol{max})}$-

Finally, it is possible to calculate the share of the enterprise *A* defective products, at which.$\Delta_{\boldsymbol{AB}} = \mathbf{0}$From relation (33) we obtain

$$\boldsymbol{y}_{\text{б}}{}^{\mathbf{0}} = \frac{\boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Вб}}}{(\boldsymbol{C}_{\text{Ая}} - \boldsymbol{C}_{\text{Аб}})(\boldsymbol{C}_{\text{Вя}} - \boldsymbol{C}_{\text{Вб}} - \boldsymbol{S}_{\text{Ая}} - \boldsymbol{S}_{\text{Аб}})} \boldsymbol{S}_{\text{к}}, \qquad (20)$$

that is, if the enterprise *A* works together with the enterprise *B* with this share of the defective products, then the total profit of the enterprises *A* and *B* does not increase, compared to the total, when they work separately, and the share of the enterprise *A* defective products is equal to $\boldsymbol{y}_{\text{б}}{}^{*}$. Obviously, $\boldsymbol{y}_{\text{б}}{}^{\mathbf{0}} > \boldsymbol{y}_{\text{б}}{}^{*}$.

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