DOI: https://doi.org/10.36059/978-966-397-363-0-11

MATHEMATICAL MODEL OF THE DYNAMIC INTERACTION OF PRODUCT PRICES IN RELATED MARKETS: THE PROBLEM OF LIMIT CYCLES

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In the context of globalization, any shifts occurring in national economies become synchronous and interconnected. Globalization processes are manifested in the intensification of foreign economic relations, the development of financial cooperation, increased labor migration, etc. As a consequence, economic recovery or, conversely, crisis phenomena in one country, to one degree or another, affect the economies of other countries. One of the main channels for synchronizing economic processes is international trade. Changes in this area can become the trigger that, as a result of a subsequent chain reaction, will ensure the spread of both positive and negative economic phenomena. That is, the changes taking place in the global economy are increasingly systemic in nature, and individual countries are elements of this system.

The presence of resonant phenomena in an open economic system determines scientific interest in the use of system dynamics methods to study possible trajectories of development of such a system as a whole, as well as individual elements of this system, taking into account both external influences and endogenous factors. One of the important directions in this sense is the study of limit cycles. The purpose of this paper is to construct a mathematical model of economic dynamics that would allow a qualitative analysis (according to phase trajectories) of the processes that determine the state of equilibrium of prices for the same product in adjacent markets.

A special case of this problem has already been considered by the authors in the paper [1]. Here we propose an alternative way to solve the general case of such a problem. As rightly stated in [2], the construction of a mathematical model of such problems is a fundamental, inevitable, but difficult and delicate process.

Periodic processes in economics can be described in mathematical language using the concept of a limit cycle [3]. Recently, the study of the properties of limit cycles has attracted significant interest, which goes beyond the scope of pure mathematics and finds application not only in economics, but also in physics, chemistry, and biology. There are two possible approaches to studying this phenomenon: the use of differential equations to describe continuous processes or difference equations if the process is discrete in time.

A mathematical model of economic dynamics for the case of two commodity markets can be represented as a system of two ordinary differential equations:

$$\begin{cases} p_1' = F_1(p_1, p_2); \\ p_2' = F_2(p_1, p_2), \end{cases} \tag{1}$$

where $p_1(t)$ and $p_2(t)$ there are prices of goods in the corresponding markets, continuously changing over time (differentiation is carried out by time);

$F_1(p_1, p_2)$, $F_2(p_1, p_2)$ are functions of excess demand for goods in each of the markets, which are nonlinear of degree no more than two.

To analyze system (1), we choose the following type of excess demand functions:

$F_1(p_1, p_2) = \lambda p_1 - p_2 + kp_1^2 + mp_1p_2 + np_2^2$, $F_2(p_1, p_2) = p_1 + ap_1^2 + bp_1p_2$,

where a, b, λ, k, m, n are some constant parameters. Then the system has the form:

$$\begin{cases} p_1' = \lambda p_1 - p_2 + kp_1^2 + mp_1p_2 + np_2^2; \\ p_2' = p_1 + ap_1^2 + bp_1p_2. \end{cases} \tag{2}$$

Note that in system (2) there are two equilibrium positions. These are the points $p_1^* = 0;\ p_2^* = 0$ and $p_1^* = 0;\ p_2^* = 1/n$, which, for certain parameter values, are focuses.

After carrying out algebraic transformations with the introduction of new variables, we obtain a quadratic system of the form:

$$\begin{cases} x' = -y + a_{20}x^2 + a_{11}xy + a_{02}y^2; \\ y' = x + b_{20}x^2 + b_{11}xy + b_{02}y^2. \end{cases} \tag{3}$$

System (3) is a special case of a six-parameter system of two differential equations, which is presented in Poincaré normal form. The coefficients of this system are determined by the following relations:

$$a_{20} = -\frac{7}{3}n;\ a_{11} = 5a_0;\ a_{02} = n;\ b_{20} = a_0;\ b_{11} = -2n;\ b_{02} = 0, \tag{4}$$

where $a_0 = a$ if $p_1^* = 0;\ p_2^* = 0$ or $a_0 = -a$ if $p_1^* = 0;\ p_2^* = 1/n$.

In accordance with the fundamental work [4], qualitative analysis (based on phase trajectories) consists in sequentially determining the values of three such quantities:

$$V_1 = \alpha A - \beta B; \tag{5}$$

$$V_2 = \left(\beta\left(5A - \beta\right) + \alpha\left(5B - \alpha\right)\right)\gamma,\ \text{if}\ \ V_1 = 0; \tag{6}$$

$$V_3 = (\beta A + \alpha B)\gamma\delta,\ \text{if}\ \ V_1 = V_2 = 0. \tag{7}$$

This method allows us to work only with real numbers. Note that, up to a constant factor, the quantities V_1, V_2, and V_3 correspond to the first three Lyapunov quantities l_1, l_2, and l_3.

To simplify the calculations, we introduce auxiliary coefficients:

$$\begin{gathered} A = a_{20} + a_{02};\ B = b_{20} + b_{02};\ \alpha = a_{11} + 2b_{02};\ \beta = b_{11} + 2a_{20} \\ \gamma = b_{20}A^3 - (a_{20} - b_{11})A^2B + (b_{02} - a_{11})AB^2 - a_{02}B^3 \\ \delta = a_{02}^2 + b_{20}^2 + a_{02}A + b_{20}B. \end{gathered} \tag{8}$$

Comparing relations (4) and (8), we obtain:

$$A = -\frac{4}{3}n;\ B = a_0;\ \alpha = 5a_0;\ \beta = -\frac{20}{3}n;$$

$$\gamma = a_0\left(-\frac{4}{3}n\right)^3 - \left(\frac{1}{3}n\right)\left(-\frac{4}{3}n\right)^2 a_0 + (-5a_0)\left(-\frac{4}{3}n\right)a_0^2 - na_0^3 =$$

$$= -\frac{16}{9}n^3a_0 + \frac{17}{3}na_0^3 = \frac{na_0}{9}\left(51a_0^2 - 16n^2\right)$$

$$\delta = n^2 + a_0^2 - \frac{4}{3}n^2 + a_0^2 = \frac{1}{3}\left(6a_0^2 - n^2\right).$$

Now we find sequentially the values of each of the quantities V_1, V_2, and V_3.

1) From relation (5) we obtain:

$$V_1 = A\alpha - B\beta = \left(-\frac{4}{3}n\right)5a_0 - a_0\left(-\frac{20}{3}n\right) = 0.$$

2) Since we received that $V_1 = 0$, then from relation (6) we find V_2:

$$V_2 = \left(\beta\left(5A - \beta\right) + \alpha\left(5B - \alpha\right)\right) =$$

$$= \left(-\frac{20}{3}n\left(-\frac{20}{3}n + \frac{20}{3}n\right) + 5a_0\left(5a_0 - 5a_0\right)\right) = 0.$$

3) Since such equalities hold $V_1 = V_2 = 0$, then from (7) we find the value V_3:

$$V_3 = \left(A\beta + B\alpha\right)\gamma\delta = \left(\left(-\frac{4}{3}n\right)\left(-\frac{20}{3}n\right) + 5a_0^2\right)\frac{na_0}{9}\left(51a_0^2 - 16n^2\right)\frac{1}{3}\left(6a_0^2 - n^2\right) =$$

$$= \frac{na_0}{27}\left(\frac{80}{9}n^2 + 5a_0^2\right)\left(51a_0^2 - 16n^2\right)\left(6a_0^2 - n^2\right).$$

Considering that $a_0^2 = a^2$, we get the value V_3:

$$V_3 = \frac{na}{27}\left(\frac{80}{9}n^2 + 5a^2\right)\left(51a^2 - 16n^2\right)\left(6a^2 - n^2\right)$$

Let us recall that the parameters V_1, V_2, and V_3 coincide with the values of the first three Lyapunov quantities l_1, l_2, and l_3. up to a constant factor. We find that for all rational values of the parameters the following statement is true: $V_3 \neq 0$. Thus, for both focuses we have the following results: $V_1 = V_2 = 0$ and $V_3 \neq 0$.

It follows that around each of the two foci (equilibrium positions) there are three limit cycles, respectively, a 3:3 ratio holds. Note that the obtained result is consistent with the results for the "predator-prey" model obtained from studying the evolution of innovation processes [5].

The proposed mathematical model of economic dynamics allows us to identify the presence of cyclic processes in a small neighborhood near equilibrium positions. Further research will be aimed at studying the nature of these limit cycles, which will allow us to identify unstable dynamic regimes and avoid the occurrence of bifurcations and catastrophes leading to instability in interconnected markets of objects of any nature, which can act as a product.

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